Grade 6 Math Circles
March 26-28, 2024
Measurement and Number Systems

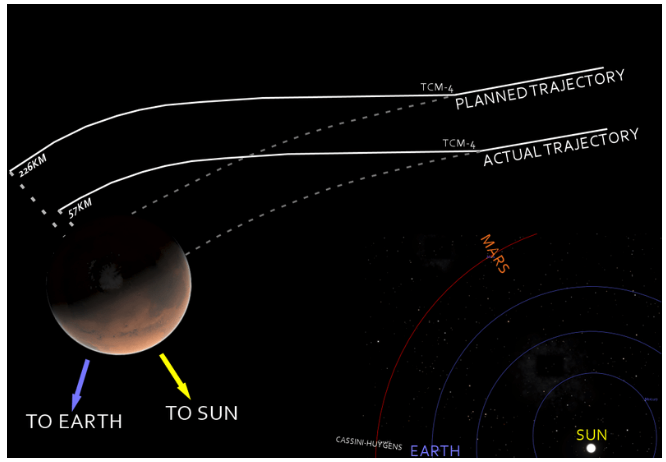
Introduction

Measurement is seen all over our daily lives: mathematics, sciences, laboratories, cooking, sports, art, farming — essentially everywhere. It dates back to the 4^{th} millennium BC, since even the earliest civilizations needed measurement systems for agriculture, trading, hunting, and more.

Systems of measurement have deep historical significance because they are entirely human-made. No god or law of physics told humans to use a particular measurement, so the systems that different groups used throughout history reflects how they would live. For example, people in the Stone Age often used limbs for measurement, so when building a shelter, they may have used the length of their arm to mark the distance between two trees, then cut a branch to fit this distance. Now, humans often use lasers for exact measurements of distance in the decimal system. Measurement systems evolve with technology and ways of living.

Mars Climate Orbiter

In 1999, NASA sent the Mars Climate orbiter into space to investigate the climate and surface of Mars. Even though their math was correct, the engineers mixed two measurement systems: US customary and metric. Calculations were done in metres, feet, miles and kilometers, but by failing to account for unit conversions, the mission resulted in a massive explosion of the $193 million USD spacecraft.



Common Measurement Systems

The two measurement systems that NASA confused in their calculations were US customary and metric. The metric system is a measurement system that most countries (including Canada) officially use. It is decimal-based (or 'base 10') and has units like metres, litres, and grams. The metric system is also known as the International System of Units, sometimes called 'SI' for Le Système International.

US Customary is a measurement system that is unique to the United States and includes units like feet, gallons, and ounces. There are many other measurement systems that people use in different countries and industries, however these are the two most common systems.

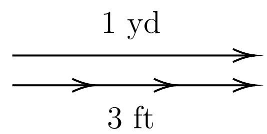
Units Conversions

Unit conversions are simply methods of starting with one unit system and ending up with another. There are many reasons why we may want to do this. Consider an engineer at Tesla calculating the maximum acceleration of a new electric vehicle. If they had some calculations with pounds and some with kilograms, their calculations would be incorrect. Or, perhaps we wish to communicate our scientific findings to someone across the world who uses different units from us. Even if our findings are correct, if they are unable to convert these units to their own, our findings are useless.

Conversion Factors

A conversion factor is a single multiplication (factor) that takes us from one unit to another; it is what makes unit conversions possible. A simple example is converting from feet to yards. There are exactly 3 feet in 1 yard, so the conversion factor looks like this:

$$\frac{3 \text{ ft}}{1 \text{ yd}}$$



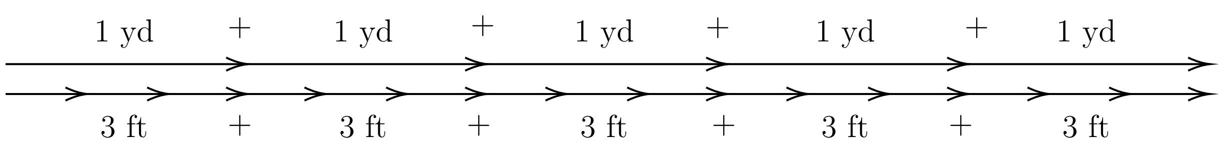
Depending on which unit we are converting from, we can also flip this fraction to

$$\frac{1 \text{ yd}}{3 \text{ ft}}$$

The important thing is that **the unit we want to get rid of is always on the bottom of the fraction!** This is because these units will 'cancel' each other out, similar to how the '2's cancel in $\cancel{2} \times \frac{1}{\cancel{2}} = 1$. Let's say we want to determine how many feet are in 5 yards. The conversion factor must have the 'yards' unit on the bottom of the fraction so that the yards cancel and we are left with feet. The calculation is:

$$5 \text{ yd} = 5 \cancel{\text{yd}} \times \frac{3 \text{ ft}}{1 \cancel{\text{yd}}} = 5 \times 3 \text{ ft} = 15 \text{ ft}$$

This equation tells us that 5 yards is exactly equivalent to 15 feet.



This may be a simple calculation, but the way we use unit conversions is often much more complicated. Note that all conversion factors are equal to 1. In the example above, when we multiplied by $\frac{3 \text{ ft}}{1 \text{ yd}}$, we are technically multiplying by 1 since 3 feet is *equal* to 1 yard. Therefore (physically) $\frac{3 \text{ ft}}{1 \text{ yd}} = 1$.

Example 1

How many seconds are in twelve and a half minutes?

Solution:

The quantity we are starting with is 12.5 minutes, and we want to somehow get to a number of seconds. This means that the bottom of our conversion factor must have units of minutes. We recall that there are 60 seconds in 1 minute, so our conversion factor is $\frac{60 \text{ s}}{1 \text{ min}}$. Multiplying this with our starting value of 12.5 min, we see

$$12.5 \text{ min} = 12.5 \cancel{\text{min}} \times \frac{60 \text{ s}}{1 \cancel{\text{min}}} = 12.5 \times 60 \text{ s} = 750 \text{ s}$$

Exercise 1

How many hours are in 1,218 minutes?

Multiple Conversions

In the previous section, we made simple jumps: from hours to minutes, minutes to seconds, and feet to yards. This can often be done in our head. Many of us know that in order to go from seconds to minutes, we just divide by 60. However, without keeping track of units, longer calculations can actually be *more* difficult to compute. If we divide 60 seconds by the number 60 instead of multiplying by the conversion factor $\frac{1 \text{ min}}{60 \text{ s}}$, we get 60 s = 1 s which of course is not true.

If we have a more complicated problem like finding the number of seconds in 3 decades, we likely need multiple conversion factors. For example, we could first jump from 3 decades to years:

$$3 \cancel{\text{decades}} \times \frac{10 \text{ years}}{1 \cancel{\text{decade}}} = 30 \text{ years}$$

Next, we jump from years to days:

$$30 \cancel{\text{years}} \times \frac{365 \text{ days}}{1 \cancel{\text{year}}} = 10,950 \text{ days}$$

Now days to hours:

$$10,950 \cancel{\text{days}} \times \frac{24 \text{ hours}}{1 \cancel{\text{day}}} = 262,800 \text{ hours}$$

Now hours to minutes:

$$262,800 \cancel{\text{hours}} \times \frac{60 \text{ min}}{1 \cancel{\text{hour}}} = 15,768,000 \text{ min}$$

Finally, minutes to seconds:

$$15,768,000 \cancel{\text{min}} \times \frac{60 \text{ s}}{1 \cancel{\text{min}}} = 946,080,000 \text{ s}$$

Note that we can do this entire calculation in a single line. We continuously multiply by a new conversion factor to cancel all units that are *not* seconds (our desired unit).

$$3 \cancel{\text{decades}} \times \frac{10 \cancel{\text{years}}}{1 \cancel{\text{decade}}} \times \frac{365 \cancel{\text{days}}}{1 \cancel{\text{year}}} \times \frac{24 \cancel{\text{hours}}}{1 \cancel{\text{day}}} \times \frac{60 \cancel{\text{min}}}{1 \cancel{\text{hour}}} \times \frac{60 \text{ s}}{1 \cancel{\text{min}}} = 946,080,000 \text{ s}$$

Notice that we are really multiplying by '1' for each conversion factor, since

$$\frac{10 \text{ years}}{1 \text{ decade}} = \frac{365 \text{ days}}{1 \text{ year}} = ... = 1$$

Example 2

Marlene is prescribed 60 mg tablets to take 4 times a day for 3 weeks. How many milligrams does she take in total?

Solution:

We start by realizing there are 60 mg per tablet, or $\frac{60 \text{ mg}}{1 \text{ tablet}}$. To eliminate the 'tablet' units, we multiply by $\frac{4 \text{ tablets}}{1 \text{ day}}$. Then we must eliminate the 'day' units, so we multiply by $\frac{7 \text{ days}}{1 \text{ week}}$. Finally, we multiply by our total of 3 weeks to get

$$\frac{60 \text{ mg}}{1 \cancel{\text{tablet}}} \times \frac{4 \cancel{\text{tablets}}}{1 \cancel{\text{day}}} \times \frac{7 \cancel{\text{days}}}{1 \cancel{\text{week}}} \times 3 \cancel{\text{weeks}} = 60 \text{ mg} \times 4 \times 7 \times 3 = 5,040 \text{ mg}$$

Exercise 2

Joseph spends 3 hours writing a short story. If each minute he writes an average of 17 words, how many words does he write in total?

Conversion Charts

Below are charts relating the different types of units for different types of measurements. Throughout each exercise it will be important to refer back to these charts, for they are the basis of all unit conversions.

TABLE 1: Length

Customary	Metric
1 inch	2.54 cm
1 foot	30.48 cm
1 yard	0.91 m
1 rod	5.00 m
1 mile	1.61 km
1 furlong	201.16 m

TABLE 2: Volume

Customary	Metric
1 teaspoon	4.93 mL
1 tablespoon	14.79 mL
1 fluid ounce	29.57 mL
1 cup	236.59 mL
1 pint	473.18 mL
1 gallon	3.79 L

TABLE 3: Mass

Customary	Metric
1 ounce	28.35 g
1 pound	453.59 g
1 short ton (2000 pounds)	1 metric ton

TABLE 4: Energy

Customary	Metric
1 British thermal unit	1055 J
1 watt-hour	3600 J

TABLE 5: Pressure

Customary	Metric
1 pound-force/in^2	6894.76 Pa
1 atmosphere	101325 Pa

TABLE 6: Power

Customary	Metric
1 British thermal unit/hour	0.29 W
1 horsepower	745.69 W

Exercise 3

Take a cup of volume 30 mL. Completely fill this cup with beads and count how many beads it takes to fill 30 mL. Write this value down. By comparing 30 mL to n beads, you have created a new way to measure volume. Now, take the shape you were given (unknown volume). Use your new bead measurement system to approximate the volume of the shape. Once you are told the true volume, find the percent error in your approximation by using the formula

$$\% \text{ error} = \frac{(\text{Estimate} - \text{True Value})}{\text{True Value}} \times 100\%$$

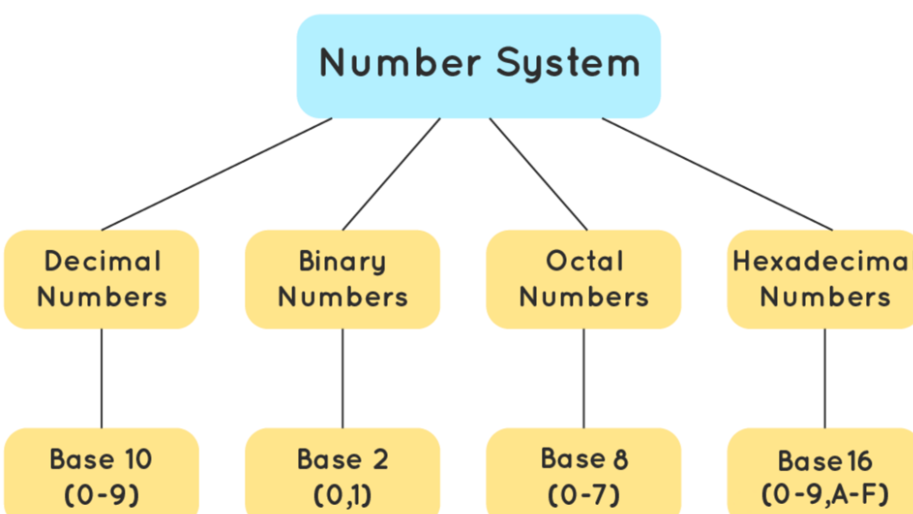
and compare with those around you. For example, if you guess 62 mL and the true value is 60 mL, the percent error is

$$\frac{(62 \text{ mL} - 60 \text{ mL})}{60 \text{ mL}} \times 100\% = 0.0333 \times 100\% = 3.33\%$$

Compare with those around you.

Number Systems

While there are many unit systems that describe the same quantities, there are also different systems to count entirely.



Decimal Counting

The number system we often use is the decimal or 'base 10' system. It is called base 10 because our measurement system (metric) is based on powers of 10. The prefix of each unit determines how large the number is. For example, a centimetre is one hundredth of a metre because the prefix 'centi' means 'hundredth'. The same goes for millimetre, kilometre, micrometre, nanometre, etc. They are all just powers of 10 different from each other.

Humans gravitate towards base 10 systems instinctively because we have 10 fingers and 10 toes. If we were octopuses with 8 legs, we would perhaps gravitate towards an octal (base 8) system of counting. If we were pigeons with 2 wings, we might choose a binary (base 2) system. The choice truly is **completely random**, and no system is better than another in general. Our counting systems are a result of our environment and evolution as a species.

Binary Counting

As seen in the figure above, binary counting is base 2 and only uses digits with 0 and 1; it is primarily used by computers. Unlike humans, octopuses, and pigeons, computers do not have any fingers or toes. The only thing it can base its counting on is electrical current by distinguishing an active electronic component (on) and an inactive one (off). The 0 and 1 digits can be thought of as a switch, where 1 represents the active component and the 0 represents the inactive one. Each of these binary digits are called 'bits'.

Before learning about how to count in binary, it is helpful to review how we count with base 10. In this system, we can work with digits 0 to 9, however once we go past 9, we do not have a unique symbol to represent each number. We must *reuse* digits 0 to 9 to represent a number like 10 or 52. Consider the number 15,732. We can represent this number equivalently as

$$(1 \times 10,000) + (5 \times 1,000) + (7 \times 100) + (3 \times 10) + (2 \times 1)$$

This process of 'reusing' is necessary to count in binary. Consider the chart below relating the two counting systems.

Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Binary	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110

TABLE 7: Decimal vs Binary

As soon as we run out of numbers to use, we are forced to add another 'bit' to the left (by convention) and restart the cycle. Converting binary to decimal can be summarized in Table 8 below. Consider the arbitrary binary number 101010.

1	**0**	**1**	**0**	**1**	**0**
$\times 32$	$\times 16$	$\times 8$	$\times 4$	$\times 2$	$\times 1$
2^5	2^4	2^3	2^2	2^1	2^0
Thirty-twos	Sixteens	Eights	Fours	Twos	Ones

TABLE 8: Binary Conversion

The position of the digit tells us the exponent on the 2. For example, the fifth column to the left multiplies the binary digit by 2^5 (or just 32). Converting this binary number to decimal, we have

$$101010 = (1 \times 32) + (0 \times 16) + (1 \times 8) + (0 \times 4) + (1 \times 2) + (0 \times 1) = 32 + 8 + 2 = 42$$

Notice that the 0 digits where the switch is "off" force the term to disappear. The only surviving terms are those with a 1 digit where the switch is "on".

Exercise 4

(a) How many bits is the binary number 110101001?

(b) What is its decimal representation?

Example 3

What is the largest decimal number we can represent with 8 bits? 9 bits? 10 bits? Explain the pattern.

Solution:

We create the largest number possible by making sure each "switch" is turned on. 8 bits means we have 8 binary digits. If each of these digits are 1, their switch is turned on and they create the largest number possible. Therefore, the binary number 11111111 gives us the largest possible decimal representation. Its decimal value is

$$(1 \times 2^7) + (1 \times 2^6) + (1 \times 2^5) + (1 \times 2^4) + (1 \times 2^3) + (1 \times 2^2) + (1 \times 2^1) + (1 \times 2^0)$$

$$= (1 \times 128) + (1 \times 64) + (1 \times 32) + (1 \times 16) + (1 \times 8) + (1 \times 4) + (1 \times 2) + (1 \times 1)$$

$$= 128 + 64 + 32 + 16 + 8 + 4 + 2 + 1$$

$$= 255$$

Notice this is one less than $2^8 = 256$, but once we introduce another digit to the left (the 9^{th} digit), we allow a new 2^8 possible decimal numbers. Therefore, the largest 9-bit number is 111111111, which has a decimal representation $255 + 2^8 = 255 + 256 = 511$. Notice $511 = 2^9 - 1$. The largest 10-bit number is 1111111111, which has a decimal representation $511 + 2^9 = 511 + 512 = 1023$. Notice $1023 = 2^{10} - 1$. We see that to find the next largest decimal value, we just add $1023 + 2^{10} = 1023 + 1024 = 2047$. We can generalize this:

$$\boxed{\text{For an } n\text{-bit number, its largest decimal value is } 2^n - 1.}$$

Other Number Systems

We saw that decimal and binary counting systems used digits 0-9 and 0 & 1, respectively. There are many other systems we can use as well. For example the hexadecimal, or 'hex', system is technically base 16, but it is a ***bit*** different! Instead of using numbers 0-15, it only uses 10 numbers (0-9) and 6 letters (A-F). Hex is used most commonly in RGB colouring, which is assigning every possible colour a six-digit code that can be broken down into red, green and blue components using the hexadecimal system.

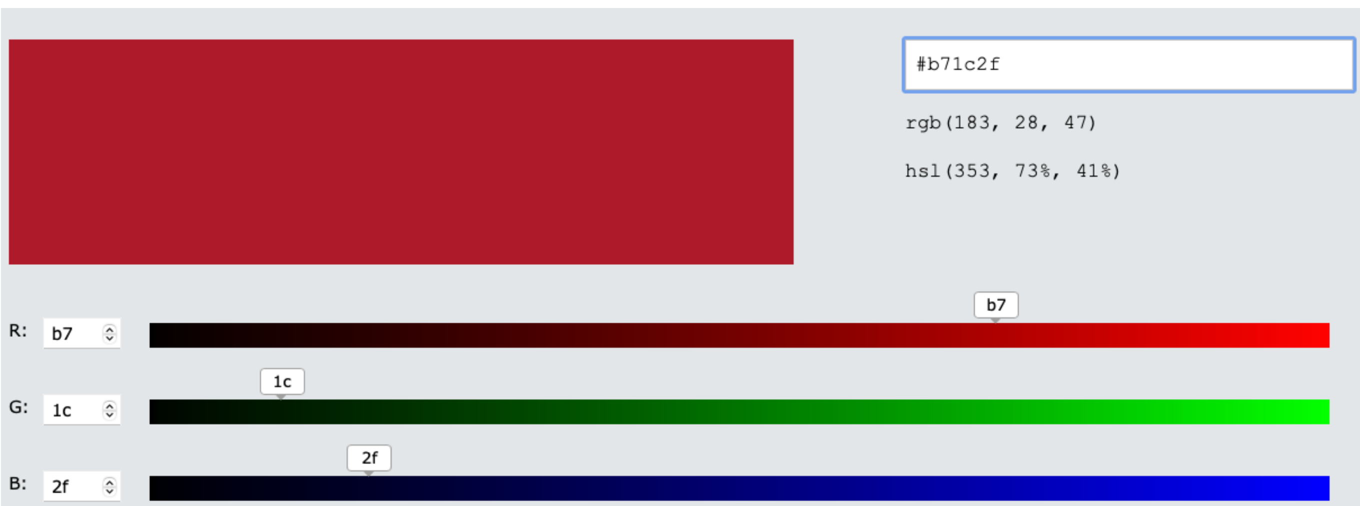
Imagine we have a colour represented by the code B71C2F. Splitting into groups of 2, we see

$$\text{B71C2F} \rightarrow \text{B7 \& 1C \& 2F}$$

Converting into decimal, it is true that B7=183, 1C=28 and 2F=47. The first two numbers represent how red the colour is, the middle two numbers represent how green the colour is, and the last two numbers represent how blue it is. We must divide each of the decimal numbers by 255, since that is the maximum value a two-digit hex number can be. The numbers then become

$$\frac{183}{255}, \frac{28}{255}, \frac{47}{255} \rightarrow 0.72,\ 0.11,\ 0.18$$

Essentially, this means the colour is 72% its full redness, 11% its full greenness, and 18% its full blueness. Here is what it looks like, for reference:



Using the hexadecimal system, we are able to represent nearly $256^3 \simeq 16.8$ million distinct colours with just six digits!

Exercise 5

Given the hex number '00' converts to 0, and 'FF' converts to 255 in decimal form, what might the hex code 000000 represent? What about FFFFFF?

Some other types of number systems are octal (base 8) and logarithmic, which is a bit trickier. The point is that there are many different number systems; some are useful in computer graphics and electronics, while others are useful experimentation and mental math. They all serve a purpose!